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HESSENBERG VARIETIES

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ABSTRACT. Numerical algorithms involving Hessenberg matrices correspond to dynamical systems which evolve on the subvariety of complete flags $S_1 \subset S_2 \subset \cdots \subset S_{n-1}$ in $\mathbb{C}^n$ satisfying the condition $s(S_i) \subset S_{i+1}$, $\forall i$, where s is an endomorphism of $\mathbb{C}^n$. This paper describes the basic topological features of the generalization to subvarieties of G/B, G a complex semisimple algebraic group, which are indexed by certain subsets of negative roots. In the special case where the subset consists of the negative simple roots, the variety coincides with the torus embedding associated to the decomposition into Weyl chambers.

1. INTRODUCTION

In this paper we study the basic topological features of certain subvarieties of flag manifolds. The original interest in these varieties, a subclass of which was investigated in [5–7], comes from the study of Hessenberg and banded forms for matrices. We omit a discussion of the applications here and refer the interested reader to these papers and the references contained therein.

If G is a semisimple algebraic group over $\mathbb{C}$ and B a Borel subgroup, with Lie algebras $\mathbf{g}$ and $\mathbf{b}$ respectively, define a *Hessenberg space* H to be a $\mathbf{b}$-submodule of $\mathbf{g}$ containing $\mathbf{b}$. If we fix an element $s \in \mathbf{g}$, the subset of G consisting of those elements g for which $\mathrm{Ad}(g^{-1})(s) \in H$ is a union of cosets gB. We therefore obtain a subvariety $X_H(s)$ of G/B on which the centralizer of s in G acts naturally. In our work we consider the special case in which s is a regular semisimple element in the Lie algebra of a maximal torus $T \subset B$. In this case, the centralizer of s in G is the torus T itself and we can apply the theory of Bialynicki-Birula [1, 2]. In particular, we obtain the Betti numbers of $X_H(s)$ by means of interesting combinatorial functions on the root system associated to the previous data. We do not deepen the analysis of the combinatorial aspects of the results, but we observe that in the special case in which H is obtained by adding to $\mathbf{b}$ all the root-spaces corresponding to the negative simple roots, the variety $X_H(s)$ coincides with the torus embedding associated to the decomposition of $\hom_{\mathbb{Z}}(X(T), \mathbb{R})$ ($X(T)$ the character group) into Weyl chambers. If $G = SL(n, \mathbb{C})$, our calculation shows that the even Betti numbers of $X_H(s)$ are given by the classical Eulerian numbers.

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A construction similar to ours is considered in [3], where the author is interested in the variety of real symmetric isospectral tridiagonal matrices. This variety is in fact the real analogue of the (height one) Hessenberg variety as studied in [6] for $G = SL(n, \mathbb{C})$.

We adopt the notations and terminology of the theory of algebraic groups. For standard definitions and results we refer to [9, 12].

II. Definitions and basic properties

Let G denote a semisimple algebraic group over $\mathbb{C}$, B a Borel subgroup, T a maximal torus in B, N the normalizer of T and $W = N/T$ the Weyl group. Let $\mathbf{g}$, $\mathbf{b}$, $\mathbf{t}$ be the Lie algebras of G, B, T, and let Φ^+, Φ^-, Δ be the positive, negative and simple roots associated with the previous data. We define two sets:

$\mathscr{H} = \{\text{subspaces } H \text{ of } \mathbf{g} | \mathbf{b} \subset H \text{ and } H \text{ is a } \mathbf{b}\text{-submodule}\}$,

$\mathscr{M} = \{\text{subsets } M \subset \Phi^- | \text{ if } \beta \in M,\ \alpha \in \Delta, \text{ and } \beta + \alpha \in \Phi^-, \text{ then } \beta + \alpha \in M\}$.

Lemma 1. *There is a one-to-one correspondence between* $\mathscr{H}$ *and* $\mathscr{M}$ *given by*

$$H = \mathbf{b} \oplus \sum_{\alpha \in M} \mathbf{g}^\alpha ,$$

where $\mathbf{g}^\alpha$ *is the root space for* α.

Proof. Straightforward. □

Let $s \in \mathbf{t}$ be a regular semisimple element and let $H \in \mathscr{H}$. Set

$$G_H(s) = \{g \in G | g^{-1} \cdot s \in H\},$$

where $g \cdot x$ denotes $\mathrm{Ad}(g)(x)$. $G_H(s)$ is clearly a subvariety of G since the map $g \mapsto g^{-1} \cdot s$ is a morphism and H is a linear space. Also, $G_H(s)$ is a union of cosets gB, so we set

$$X_H(s) = \{gB \in G/B | g \in G_H(s)\}.$$

For the remainder of the paper, we suppress the dependence on s in the notation.

Example. For p a nonnegative integer, define

$$M_p = \{\beta \in \Phi^1 | \mathrm{ht}(-\beta) \le p\},$$

where "ht" stands for height. If $G = SL(n, \mathbb{C})$ and B and T are as usual, then the subspace H_p of $sl(n, \mathbb{C})$ corresponding to M_p consists of those matrices (h_{ij}) for which $h_{ij} = 0$ if $i - j > p$, i.e., in generalized Hessenberg-type banded form. This explains the origin of the term "Hessenberg variety." The variety $X_{H_p}(s)$ can be viewed as the complete flags $S_1 \subset S_2 \subset \cdots \subset S_{n-1}$ in $\mathbb{C}^n$ which satisfy the condition $s(S_i) \subset S_{i+p}$, $\forall i$.

Proposition 2. X_H *is a* T*-stable subvariety of* G/B.

Proof. X_H is a subvariety of G/B in fact isomorphic to G_H/B. If $g \in G_H$ and $t \in T$, we have $(tg)^{-1} \cdot s = g^{-1} \cdot (t^{-1} \cdot s) = g^{-1} \cdot s \in H$. □

Proposition 3. X_H *contains* $(G/B)^T$ *(the fixed points of* T *in* G/B*).*

Proof. $(G/B)^T = NB/B$ and if $n \in N$, we have $n^{-1} \cdot s \in \mathbf{t} \subset H$. □

Next we recall some standard facts on torus actions.

Lemma 4. *Let X be a complete variety with an action of a torus T and X^T be the fixed points. If every $x \in X^T$ is a smooth point of X, then X is smooth.*

Proof. The singular points of X form a closed T-stable subvariety. Now any complete variety with a T action has fixed points (the closed T-orbits), so if X is smooth in X^T it is smooth everywhere. □

Lemma 5. *The dimension of X (as in the previous lemma) is the maximum of the dimensions of X in the points of X^T.*

Proof. Each irreducible component of X is complete and hence contains a fixed point. In this point, the dimension is equal to the maximum dimension of the components containing it. □

Theorem 6. *The variety X_H is smooth equidimensional of dimension equal to* $\dim H/\mathbf{b}$.

Proof. Consider the projection $\pi\colon G \to G/B$ and $G_H \to X_H$. Since π is smooth and $G_H = \pi^{-1}(X_H)$, it is sufficient to show that G_H is smooth equidimensional of dimension $\dim H$. In fact, it is enough to prove this in the points $n \in N$, by the previous lemmas. Now let $n \in N \subset G_H$. Consider the map $\rho\colon G \to \mathbf{g}$ given by $\rho(g) = g^{-1} \cdot s$. This is again a smooth fibration (since it is an orbit map) with image the G-orbit of s. Since $T = \{g \in G | g^{-1} \cdot s = s\}$, we have that the fibers of the map ρ have dimension $\dim \mathbf{t}$. Now, $G_H = \rho^{-1}(H \cap G \cdot s)$, so it is enough to show that $H \cap G \cdot s$ is smooth in the point $n^{-1} \cdot s$ and of dimension $\dim H - \dim \mathbf{t}$. In order to prove these statements, it is enough to prove that $H + [\mathbf{g}, (n^{-1} \cdot s)] = \mathbf{g}$, since then the intersection of H and $G \cdot s$ is transversal in $n^{-1} \cdot s$ ($[\mathbf{g}, (n^{-1} \cdot s)]$ is the tangent space to the orbit $G \cdot s$ in $n^{-1} \cdot s$) and so $H \cap G \cdot s$ is smooth in $n^{-1} \cdot s$ of dimension $\dim H + \dim([\mathbf{g}, n^{-1} \cdot s]) - \dim \mathbf{g} = \dim H - \dim \mathbf{t}$. The fact that $H + [\mathbf{g}, (n^{-1} \cdot s)] = \mathbf{g}$ is trivial, since $n^{-1} \cdot s$ is a regular element in $\mathbf{t}$ and so $[\mathbf{g}, (n^{-1} \cdot s)] = \mathbf{n}_+ \oplus \mathbf{n}_-$ ($\mathbf{n}_+ = \sum_{\alpha \in \Phi^+} \mathbf{g}^\alpha$) and since $\mathbf{t} \subset H$, we are done. □

III. Betti numbers

We apply the theory of Bialynicki-Birula [1, 2] to our situation. $X_H \subset G/B$ is T-stable and $X_H^T = (G/B)^T$ is in one-to-one correspondence with N/T, hence indexed by the Weyl group W. If $w \in W$, the point $p_w \in (G/B)^T$ is nB/B, where $w = nT/T$.

We choose a regular one-parameter subgroup $\lambda\colon G_m \to T$ defining the fundamental Weyl chamber. I.e., if $\alpha\colon T \to G_m$ is a simple root, we have $(\alpha \circ \lambda)(t) = t^{\langle \alpha | \lambda \rangle}$ and $\langle \alpha | \lambda \rangle > 0$. Then $(G/B)^T = (G/B)^{\lambda(G_m)}$ and the Bruhat decomposition of G/B is just the plus decomposition of Bialynicki-Birula, i.e.,

$$C_w^+ = \left\{ x \in G/B | \lim_{t \to 0} \lambda(t) x = p_w \right\}.$$

By the same theory one has a cell decomposition of X_H into cells

$$\overline{C}_w^+ = \left\{ x \in X_H | \lim_{t \to 0} \lambda(t) x = p_w \right\}.$$

The cells $\overline{C}_w^+$ are isomorphic to affine spaces and their closures are a basis for the Chow group of X_H, which is isomorphic to the cohomology of X_H.

In particular, there is no torsion in homology, no odd homology and the Betti number b_{2k} is the number of cells $\overline{C}_w^+$ of (complex) dimension k.

To compute the dimension of $\overline{C}_w^+$, $w = nT/T$, one looks at the tangent space $T_{nB}(X_H)$ of X_H in $p_w = nB/B$. This is a representation of $\lambda(G_m)$ since p_w is an isolated fixed point, and it decomposes into positive and negative weights:

$$\dim \overline{C}_w^+ = \#\{\text{positive weights of } \lambda(G_m) \text{ in } T_{nB}(X_H)\}\,.$$

Finally, let us calculate $T_{nB}(X_H)$ as a representation.

Lemma 7. *$T_{nB}(X_H)$ is isomorphic to $nH/n\mathbf{b}$ as a T-module.*

Proof. Let $l_{n^{-1}}: G \to G$ be the map $g \mapsto n^{-1}g$. It gives by differentiation an isomorphism of $T_n(G)$ with $T_1(G) = \mathbf{g}$. We want to compute the image of $T_n(G_H)$ in $\mathbf{g}$ under this isomorphism. Clearly $T_n(G_H)$ is mapped to $T_1(n^{-1}G_H)$. Now $n^{-1}G_H = \{x \in G | x^{-1}n^{-1}\cdot s \in H\}$. Hence $T_1(n^{-1}G_H)$ is the preimage of H under the map of tangent spaces $d\pi: \mathbf{g} \to \mathbf{g}$, where $\pi: G \to \mathbf{g}$ is the map $\pi(x) = x^{-1}n^{-1}\cdot s$. Now $d\pi(x) = [x, n^{-1}\cdot s]$ and since $n^{-1}\cdot s$ is a regular semisimple element in $\mathbf{t}$, we have that $[x, n^{-1}\cdot s] \in H$ if and only if $x \in H$. Therefore $T_n(G_H)$ is identified with H under the map $(dl_{n^{-1}})_n$.

We need to see how this identification behaves under T. Consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{l_{n^{-1}}} & G \\ \Big\downarrow{\scriptstyle l_t} & & \Big\downarrow{\scriptstyle l_{n^{-1}tn}} \\ G & \xrightarrow{l_{n^{-1}}} & G \end{array}$$

The T-action on $T_n(G_H)$ becomes, under the above isomorphism, the action of T on H given by $t\cdot h = (n^{-1}tn)h$ (twisted action by n^{-1}). Moreover, the tangent space of nB is mapped to the tangent space of B, i.e., $\mathbf{b}$. Since $T_{nB}(X_H) = T_n(G_H)/T_n(nB)$, the representation of T on $T_{nB}(X_H)$ is identified with $H/\mathbf{b}$ by the twisted action. In other words, consider the linear isomorphism $H \to nH$ (multiplication by n). The twisted action of T on nH becomes the usual adjoint action of T on nH and so the space $T_n(X_H)$ is identified with $nH/n\mathbf{b}$ as a T-module. □

Theorem 8. *For every $w \in W$, let $e_w^M = \#\{\alpha \in M | w(\alpha) \in \Phi^+\}$. The Poincaré series of X_H is $\sum_{w\in W} t^{e_w^m}$.*

Corollary 9. (i) *X_H is connected if and only if for every $w \in W$, $w \neq 1$, we have $w(M) \not\subset \Phi^-$.*

(ii) *For $H = H_p$ associated with $M_p = \{\beta \in \Phi^- | \mathrm{ht}(-\beta) \leq p\}$, $p \geq 1$, X_H is connected.*

Proof. (i) In fact the number of connected components of X_H is equal to the number of w's such that $w(M) \subset \Phi^-$. (ii) Clear. □

Remark. Let $G = SL(n, \mathbb{C})$ and identify W with the symmetric group $\Sigma(n)$ on n letters. Choose $p = 1$ and consider $H = H_1$ associated with M_1—i.e., the space of zero trace and height one Hessenberg matrices. Then, if $\sigma \in \Sigma(n)$, we have

$$e_\sigma^{M_1} = \#\{i | 1 \leq i \leq n-1\,,\ \sigma(i) > \sigma(i+1)\}\,,$$

i.e., the number of descents in the permutation σ. Therefore the even Betti numbers of X_H coincide with the classical Eulerian numbers.

IV. Identification of the variety associated with the simple roots

We will show that this variety is the torus embedding associated to the decomposition Weyl chambers. This is a variety which appears in several contexts. (See for instance [4, 8].)

Lemma 10. *Let X be a smooth connected complete variety with the action of a torus T. Let $p \in X^T$ be a fixed point. Assume that the characters of T in the tangent space $T_p(X)$ appear with multiplicity one and form a basis of the character group of T. Then there is an open T-orbit in X, and hence X is a torus embedding.*

Proof. Let $n = \dim T$ and $x_1, \dots, x_n$ the characters in the tangent space $T_p(X)$. (Thus, $\dim X = n$.) Let $\lambda\colon G_m \to T$ be a one-parameter subgroup such that $\langle x_i | \lambda \rangle > 0$, $1 \le i \le n$. We have by the theory of Bialynicki-Birula an open cell C centered in p, which is T-isomorphic to the linear representation of T in $\mathbb{C}^n$ given by the n-characters x_i. Since these characters are a basis of the character group of T, it is clear that the elements of $\mathbb{C}^n$ with all the coordinates different from zero form an orbit isomorphic to T. □

Let us now consider the set $M_1 = -\Delta$ and the corresponding Hessenberg variety which we denote by X_1. Our analysis shows that the hypotheses of Lemma 10 apply to X_1. Thus, X_1 is a torus embedding. Moreover, the T-fixed points of X_1 are in one-to-one correspondence with the elements of the Weyl group W and in a point corresponding to w, the characters of the tangent space are $w(\Delta)$. To a complete smooth torus embedding is associated a decomposition of $V := \hom_{\mathbb{Z}}(X(T), \mathbb{R})$ ($X(T)$ the character group) into polyhedral simplicial cones. The open polyhedron associated with a fixed point p is the dual cone of the set of characters in the tangent space of p. Thus we have

Theorem 11. *The variety X_1 is the torus embedding associated with the decomposition of V in Weyl chambers.*

Corollary 12. *X_1 can be endowed with a natural action of the normalizer of T.*

The cohomology ring of this variety has been analyzed by several authors. (See [10, 11].)

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